Markov chains and prediction

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Markov chains: a quick intro

- We are interested in predicting weather, and for the purposes of this example, weather can take on one of three values: \{sunny, rainy, cloudy\}.

- The weather on a given day is dependent only on the weather on the previous day. 

\[
P(w_t \mid w_{t-1}, \ldots, w_1) = P(w_t \mid w_{t-1})
\]

This is the Markov property.
Components of a Markov chain

- State space $S$: \{sunny, rainy, cloudy\}
- Transition probabilities
  \[ q(x, y) = P(w_{t+1} = x \mid w_t = y), \text{ for } x, y \in S \]
- Start state: $s$ in $S$
Markov chain example

- The transition probabilities between the various states, $q(s,s')$ is called the transition kernel.

\[
\begin{bmatrix}
\text{s} & \text{r} & \text{c} \\
0.50 & 0.25 & 0.25 \\
0.50 & 0.00 & 0.50 \\
0.25 & 0.25 & 0.50 \\
\end{bmatrix}
\]

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Using Markov chains for prediction

Suppose day 1 is rainy. We will represent this as a vector of probabilities over the three values.

\[ \pi(1) = [0 \ 1 \ 0]; \]

How do we predict the weather for day 2 given \( \pi(1) \) and the transition kernel \( q \)?

From the transition kernel, we can see that the probability of day 2 being sunny is .5, and that the probabilities for being cloudy or rainy are 0.25 each.

\[ \pi(2) = \pi(1) \ast q = [0.5 \ 0 \ 0.5]; \]
Prediction (contd.)

- We can calculate the distribution of weather at time $t+1$ given the distribution for time $t$.

\[ \pi(t+1) = \pi(t) \times q \]

\[ = (\pi(t-1) \times q) \times q \]

\[ = \pi(1) \times q^t \]
Prediction

- What's the weather going to be like on the 3rd, 5th, 7th and 9th days?

\[ \pi(3) = \pi(1) * q^2 = [0.375 \ 0.25 \ 0.375] \]
\[ \pi(5) = \pi(1) * q^4 = [0.3984 \ 0.2031 \ 0.3984] \]
\[ \pi(7) = \pi(1) * q^6 = [0.3999 \ 0.2002 \ 0.3999] \]
\[ \pi(9) = \pi(1) * q^8 = [0.4 \ 0.2 \ 0.4] \]
\[ \pi(t) = [0.4 \ 0.2 \ 0.4], \text{ for all } t \geq 9 \]
A new start state

- Let the weather on day 1 be sunny.
- How does the distribution of weather change with time?

\[ \pi(1) = [1 \ 0 \ 0] \]

\[
\begin{align*}
\pi(3) &= \pi(1) \cdot q^2 = [0.4375 \ 0.1875 \ 0.375] \\
\pi(5) &= \pi(1) \cdot q^4 = [0.4023 \ 0.1992 \ 0.3984] \\
\pi(7) &= \pi(1) \cdot q^6 = [0.4001 \ 0.2000 \ 0.3999] \\
\pi(9) &= \pi(1) \cdot q^8 = [0.4 \ 0.2 \ 0.4] \\
\pi(t) &= [0.4 \ 0.2 \ 0.4], \text{ for all } t \geq 9
\end{align*}
\]
Stationary distribution

- Independent of the start state, this Markov process converges to a stationary distribution $[0.4 \ 0.2 \ 0.4]$ in the limit.
- The stationary distribution $p^*$ is the solution to the equation $p^* q = p^*$.
- This is the distribution calculated by the PageRank algorithm.
Mapping to Python

```python
q = numpy.array([[0.5, 0.25, 0.25],
                 [0.5, 0, 0.5],
                 [0.25, 0.25, 0.5]])

p = numpy.array([0, 1, 0])

for i in range(20):
    print i, p
    p = numpy.dot(p, q)
```